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ON A CERTAIN RELATIONSHIP BETWEEN EULER'S TOTIENT AND PARTITION FUNCTION

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ABSTRACT. We study the Merca's formula which connects the partition function with the Euler's totient, we show the corresponding inverse relation and we make applications to several arithmetic functions.

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1. INTRODUCTION

Merca [1–3] deduced the following expression:

$$\sum_{k=1}^{n} k\ p(n-k) = \sum_{k=1}^{n} \varphi(k) S_{n,k}, \tag{1}$$

involving the partition function and Euler's totient [4–7], where $S_{n,k}$ is the number of k's in all partitions of n.

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Similarly [1–3]:

$$p(n) = \sum_{k=1}^{n+1} \mu(k)\, S_{n+1,k}, \tag{2}$$

with the presence of the Möbius function [4, 8, 9].

The relations (1) and (2) are remarkable because they connect a function of multiplicative number theory with one of additive number theory.

The formula (1) for several values of n implies the expressions:

$$\begin{aligned} p(0) &= \varphi(1)S_{1,1}, \\ p(1) &= \varphi(1)(S_{2,1} - 2S_{1,1}) + \varphi(2)S_{2,2}, \\ p(2) &= \varphi(1)(S_{3,1} - 2S_{2,1} + S_{1,1}) + \varphi(2)(S_{3,2} - 2S_{2,2}) \\ &\quad + \varphi(3)S_{3,3}, \\ p(3) &= \varphi(1)\big(S_{4,1} - 2S_{3,1} + S_{2,1}\big) + \varphi(2)\big(S_{4,2} - 2S_{3,2} + S_{2,2}\big) \\ &\quad + \varphi(3)\big(S_{4,3} - 2S_{3,3}\big) + \varphi(4)S_{4,4}. \end{aligned}$$

In fact, here we employ the Z-transform [10, 11] to obtain the inverse relation of (1):

$$p(n) = \sum_{k=1}^{n+1} \varphi(k)\big(S_{n+1,k} - 2S_{n,k} + S_{n-1,k}\big), \quad n \geq 0 \tag{3}$$

because $S_{n,m} = 0, n < m,\ S_{r,0} = 0, r \geq 0,\ S_{t+1,t} = 1,\ t \geq 2,\ S_{j,j} = 1,\ j \geq 1$.

Merca [1] also established the property:

$$\sum_{k=1}^{n} g(k)\, p(n-k) = \sum_{k=1}^{n} f(k)\, S_{n,k} \tag{4}$$

for an arbitrary arithmetic function f and $g(n) = \sum_{d|n} f(d)$. Here we show that applying the Z-transform to (4) implies

the following identity for any prime p:
(5)
$$\sum_{r=j}^{p} a_{p-r}\, S_{r,j} = \begin{cases} 1, & j = 1,\, p \\ 0, & 2 \leq j \leq p-1 \end{cases}, \quad p = 2, 3, 5, 7, 11, ...,$$

such that
(6)
$$a_j = \begin{cases} 0, & \text{if } j \neq \frac{m}{2}(3m+1), \\ (-1)^m, & \text{if } j = \frac{m}{2}(3m+1), \end{cases} \quad m = 0, \pm 1, \pm 2, \ldots$$

2. Proof of (3)

The formula (1) can be written as a Cauchy convolution [4]:

$$\sum_{k=0}^{n} k\, p(n-k) = \sum_{k=0}^{n} \varphi(k)\, S_{n,k} \;:= q_n, \quad n \geq 0, \tag{7}$$

because $\phi(0) = 0$; besides, we have that [10]:

$$Z\{k\} = Z\{0, 1, 2, \ldots\} = \frac{z}{(z-1)^2}, \tag{8}$$

thus the Z-transform of (7) generates the relation:

$$\begin{aligned} Z\{p(n)\} = \frac{(z-1)^2}{z}\, Z\{q_n\} &= \left(z - 2 + \frac{1}{z}\right) Z\{q_n\} \\ &= Z\{q_{n+1} - 2q_n + q_{n-1}\}, \end{aligned}$$

which implies (3), *q.e.d.*

3. Applications of (4)

(a) Let f be the nontrivial Dirichlet character modulo 4 [12, 13]:

$$f(n) = \chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases}$$

$$= \begin{cases} 1, & n \equiv 1 \pmod 4, \\ -1, & n \equiv -1 \pmod 4, \\ 0, & n \equiv 0 \pmod 2. \end{cases} \tag{9}$$

with Jacobi's identity [14–16]:

$$g(n) = \sum_{d|n} \chi_4(d) = \frac{1}{4} r_2(n), \tag{10}$$

where $r_2(n)$ is the number of representations of n as a sum of two squares [12, 17, 18]. Therefore, equation (4) gives the following property:

$$\sum_{k=1}^{n} r_2(k)\, p(n-k) = 4 \sum_{k=1}^{n} \chi_4(k)\, S_{n,k}, \quad n \geq 1. \tag{11}$$

(b) Let $f = J_m$ be the Jordan totient function [4], then $g(n) = n^m$, and from (4) we have:

$$\sum_{k=1}^{n} k^m\, p(n-k) = \sum_{k=1}^{n} J_m(k)\, S_{n,k}, \tag{12}$$

which reproduces (1) if $m = 1$.

(c) The application of the Z-transform to (4) implies:

$$Z\{g(n)\} = Z\{a_n\}\, Z\left\{ \sum_{k=0}^{n} f(k)\, S_{n,k} \right\}, \tag{13}$$

where the relation

$$Z\{p(n)\} = \big(Z\{a_n\}\big)^{-1} \tag{14}$$

was used, which is generated by MacMahon's recurrence relation [19–22]:

$$\sum_{k=0}^{n} a_k\, p(n-k) = 0, \quad n \geq 1. \tag{15}$$

Hence, from (13):

$$g(n) = \sum_{d|n} f(d) = \sum_{j=0}^{n} f(j) \sum_{r=j}^{n} a_{n-r}\, S_{r,j}, \quad n \geq 1, \tag{16}$$

for any arithmetic function f. If n is a prime number, then $g(p) = f(1) + f(p)$, and from (16) the identity (5) follows immediately.

(d) Let $f = e_0$, then $g = e$, where $e(n) = 1$ for all n and

$$e_0(n) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Hence, (4) gives the interesting property:

$$S_{n,1} = \sum_{j=0}^{n-1} p(j) = -\sum_{k=2}^{n+1} \mu(k)\, S_{n+1,k} \quad \therefore \quad p(n) = S_{n+1,1} - S_{n,1}. \tag{17}$$

(e) Let $f = \mu \therefore g = e_0$, thus (4) implies (2).

(f) Let $f(n) = |\mu(n)|$, then $g(n) = 2^{\omega(n)}$ counts the square-free divisors of n, where $\omega(n)$ is the number of distinct prime divisors of n. Then from (4):

$$\begin{aligned}\sum_{k=1}^{n} |\mu(k)|\, S_{n,k} &= \sum_{k=1}^{n} 2^{\omega(k)}\, p(n-k) \\ &= \text{Total number of square-free parts} \\ &\qquad \text{in all partitions of } n.\end{aligned} \tag{18}$$

(g) Let $f(n) = I(n) = n$ and $g = \sigma$ be the sum of divisors function. Then (4) gives the known formula [20, 23]:

$$n\, p(n) = \sum_{k=1}^{n} \sigma(k)\, p(n-k) = \sum_{k=1}^{n} k\, S_{n,k}. \tag{19}$$

Remark 1. *The expressions* (1) *and* (3) *imply the identity:*
(20)

$$p(n) = n+1+\sum_{k=1}^{n} k\big(p(n+1-k)-2\,p(n-k)+p(n-1-k)\big), \quad n \geq 0.$$

Remark 2. *If we select $f(n) = d(n)$, the number of divisors of n, then* (16) *generates the relation:*

$$d(n) = \sum_{r=1}^{n} a_{n-r} \sum_{j=1}^{r} S_{r,j}. \tag{21}$$

Remark 3. *From* (2) *and* (3)*:*

$$\sum_{k=1}^{n+1} \phi(k)\big(S_{n+1,k} - 2S_{n,k} + S_{n-1,k}\big) = \sum_{k=1}^{n+1} \mu(k)\, S_{n+1,k}, \tag{22}$$

where we can employ $n = 0, 1, 2, 3, \ldots$ to obtain $\phi(m)$ in terms of $\mu(1), \mu(2), \ldots, \mu(m)$, $m = 1, 2, 3, \ldots,$ in complete harmony with the known expression [4, 14–16]:

$$\phi(m) = \sum_{d|m} \mu(d)\, \frac{m}{d}. \tag{23}$$

4. Conclusions

Our work shows that the Z-transform is a useful tool to study relations involving arithmetic functions. In particular, its application to the results obtained by Merca [1] allows us to derive the explicit property (3) for the partition function $p(n)$ in terms of Euler's totient function, as well as the identity (5) for pentagonal numbers.

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